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Nonparametric Methods in Directional Data Analysis

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1. Introduction

In many natural and physical sciences the observations are in the form of directions – directions either in plane or in three-dimensional space. Such is the case when a biologist investigates the flight directions of birds or a geologist measures the paleomagnetic directions or an ecologist records the directions of wind or water. A convenient sample frame for two-dimensional directions is the circumference of a unit circle centered at the origin with each point on the circumference representing a direction; or, equivalently, since magnitude has no relevance, each direction may be represented by a unit vector. Such data on two-dimensional directions will be called 'circular data'. Similarly the surface of a unit sphere in three-dimensions may be used as the sample space for directions in space, with each point on the surface representing a three-dimensional direction; or alternatively, such a direction may be represented by a unit vector in three-dimensions. Such data is referred to as the 'spherical data'. Also, studies on any periodic phenomena with a known period (such as circadian rhythms in animals) can be represented as circular data, for instance by identifying each cycle or period with points on the circumference, pooling observations over several such periods, if necessary.

The analysis of directional data gives rise to a host of novel statistical problems and does not fit into the usual methods of statistical analysis which one employs for observations on the real line or Euclidean space. Since there is no natural zero-direction, any method of numerically representing a direction depends on the arbitrary choice of this zero direction. It is important that the statistical analyses and conclusions remain independent of this arbitrary zero direction. Unfortunately, however, usual statistics like the arithmetic mean and standard deviation (and all the higher moments) which one employs in linear statistical analyses fail to have this required rotational invariance so that one is forced to seek alternate statistics for describing directional data. To do this, we treat each direction as a unit vector in plane or space. One computes the resultant vector, whose direction provides a meaningful measure of the average direction in unimodal populations. The length of this vector resultant measures the concentration of the data since observations closer together lead to a longer resultant.

One of the basic parametric models for unimodal directional data is called the von Mises–Fisher distribution and is discussed briefly in Section 2. This plays as prominent a role in a directional data analysis as does the normal distribution in the linear case. Sections 3 and 4 review nonparametric methods for circular (two-dimensional) and spherical (three-dimensional) data, respectively. Section 3 is considerably larger since more distribution-free methods have been developed for circular data. The reader may consult the books by Mardia (1972), Batschelet (1981) and Watson (1983) for a more complete introduction to this novel area of statistics.

2. The von Mises–Fisher model for directional data

A parametric model which plays a central role in directional data analysis is called the von Mises–Fisher distribution. In general, if $\boldsymbol{x}$ is a unit vector in p-dimensions ($p \geq 2$) or equivalently, represents a point on S_p, the surface of a unit ball in p dimensions, then the probability density of the von Mises–Fisher distribution is of the form

$$C_p(\kappa)\exp(\kappa \cdot \boldsymbol{x}'\boldsymbol{\mu}) \tag{2.1}$$

where $\kappa > 0$ is a concentration parameter and the unit vector $\boldsymbol{\mu}$ denotes the mean direction. Here the normalizing constant

$$C_p(\kappa) = \kappa^{(1/2)p-1}/[(2\pi)^{p/2} I_{p/2-1}(\kappa)] \tag{2.2}$$

where $I_r(\kappa)$ is the modified Bessel function of the first kind and order r. When $p = 2$, this density reduces to

$$f(\alpha \mid \kappa, \mu) = [2\pi I_0(\kappa)]^{-1} \exp[\kappa \cdot \cos(\alpha - \mu)] \tag{2.3}$$

where $0 \leq \alpha < 2\pi$ and $0 \leq \mu < 2\pi$ are the angles (in polar coordinates) corresponding to $\boldsymbol{x}$ and $\boldsymbol{\mu}$ in (2.1). This density was introduced by von Mises (1918) to test the hypothesis that the atomic weights are integers. When $p = 3$, Fisher (1953) studied the pdf with zero mean direction,

$$f(\alpha, \beta \mid \kappa) = \frac{\kappa}{4\pi \sinh \kappa} e^{\kappa \cos \alpha} \cdot \sin \alpha \tag{2.4}$$

where $0 \leq \alpha < \pi$ and $0 \leq \beta < 2\pi$ are the polar coordinates of $\boldsymbol{x}$. Fisher's 1953 paper is the first comprehensive treatment of the sampling distributions and statistical inference for the spherical model (2.4).

If the concentration parameter $\kappa = 0$ in (2.1), this reduces to the uniform

(isotropic) distribution on S_p. For $\kappa > 0$, this is a unimodal distribution with mode at $\boldsymbol{x} = \boldsymbol{\mu}$. Since the likelihood for a sample of n observations is given by

$$[C_p(\kappa)]^n \cdot \exp(\kappa \cdot \boldsymbol{R}'\boldsymbol{\mu})$$

where $\boldsymbol{R}$ is the vector resultant of the sample, $\boldsymbol{R}$ is a sufficient statistic for this family of distributions. The Maximum Likelihood Estimator of μ is given by $(1/|R|) \cdot \boldsymbol{R}$, where $|R|$ is the length of the resultant. The concentration parameter κ is estimated by solving the equation

$$\frac{I_1(\kappa)}{I_0(\kappa)} = \frac{|R|}{n}$$

for $p = 2$, and the equation

$$\coth \kappa - \frac{1}{\kappa} = \frac{|R|}{n}$$

for $p = 3$. Various one- and two-sample tests on the parameters can be performed using $\boldsymbol{R}$. See, for instance, Chapters 6 and 8 of Mardia (1972). One important preliminary test is to verify if indeed there is a preferred direction, i.e., H_0: $\kappa = 0$. The UMP invariant test for this is based on the length of the resultant, $|R|$, whose density under the null hypothesis of uniform distribution (or random walk model) is given (for $r > 0$) by

$$r \int_0^\infty J_0(rt) J_0^n(t)\, t \, dt$$

when $p = 2$, and

$$\frac{r}{2^{n-1}(n-2)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \langle n - r - 2j \rangle^{n-2}$$

when $p = 3$. Here $\langle x \rangle = x$ if $x > 0$ and 0 otherwise. This test of 'no preferred direction', i.e., of H_0: $\kappa = 0$ which is based on $|R|$, is known as Rayleigh's test.

3. Nonparametric methods for circular data

Though considerable statistical theory has been developed for the von Mises–Fisher distribution and to a much lesser extent for some of the other parametric models for directions, these models may not provide an adequate description of the data or the distributional information may be imprecise. For instance, information about the unimodality or axial symmetry that a particular parametric model assumes may be lacking or might be inappropriate for a given data set. The search for methods which are robust leads naturally, as in linear statistical inference, to techniques which are nonparametric or model-free. In linear inference, there are a number of considerations on which one can justify for instance an assumption of normality as for example when one deals with averages, or when the samples are large enough. Unfortunately, there is no corresponding rationale for invoking the von Mises–Fisher distribution and thus the need for model-free methods might indeed be stronger in directional data analysis.

This section will be subdivided into three subsections dealing with one-, two- and multi-sample nonparametric techniques.

3.1. *One-sample tests and the goodness-of-fit problem*

For simplicity, let us assume that the circle has unit circumference and that the circular data is presented in terms of angles $(\alpha_1, \ldots, \alpha_n)$ with $0 \leq \alpha_i < 1$ with respect to some arbitrary zero direction. Given such a random sample, one of the fundamental problems in circular data is to test if there is no preferred direction against the alternative of one (or more) preferred direction(s). Since having no preferred direction corresponds to a uniform (or isotropic) distribution, the null hypothesis to test is

$$H_0: \quad \alpha \sim \text{uniform distribution on } [0, 1)\,. \tag{3.1}$$

As in the linear case, the goodness-of-fit problem of testing whether the sample came from a specified circular distribution can also be reduced to testing uniformity on the circle.

We seek rotationally invariant tests, i.e., tests invariant under changes in zero direction as well as the sense of rotation (clockwise or anticlockwise). There are three broad groups of tests for this problem, which are described below.

(i) Tests based on sample arc lengths or spacings. If $\alpha_{(1)} \leq \cdots \leq \alpha_{(n)}$ denote the order statistics in the linear sense, the differences

$$\alpha_i^* = (\alpha_{(i)} - \alpha_{(1)}), \quad i = 2, \ldots, n, \tag{3.2}$$

form a maximal invariant. But if one defines

$$D_i = (\alpha_{(i)} - \alpha_{(i-1)}), \quad i = 1, \ldots, n, \tag{3.3}$$

with $\alpha_{(0)} = (\alpha_{(n)} - 1)$, these are the lengths of the arcs into which the sample partitions the unit circumference and are called the sample spacings. Clearly $\alpha_i^* = \sum_{j=2}^{i} D_j$. Any symmetric function of the sample spacings will have the rotational invariance property, and Rao (1969) suggested the use of such a class of spacings tests for testing H_0 in (3.1). See Rao (1976) and the references contained there. In particular the statistic

$$\tfrac{1}{2}\sum_{i=1}^{n} \left| D_i - \frac{1}{n} \right| = \sum_{i=1}^{n} \max\left(D_i - \frac{1}{n}, 0\right) \tag{3.4}$$

corresponds to the uncovered portion of the circumference when n arcs of length $(1/n)$ are placed to cover the circumference starting at each of the observations. Its exact and asymptotic distributions and a table of percentage points are given in Rao (1976) and reproduced in Batschelet (1981). Among all such symmetric test statistics, the one based on $\sum_{i=1}^{n} (D_i - 1/n)^2$, which is referred to as the Greenwood statistic, has asymptotically maximum local power. Burrows (1979), Currie (1981) and Stephens (1981) discuss computational methods for obtaining the percentage points of Greenwood's statistic. See also Rao and Kuo (1984) for a discussion of some variants of this statistic which are asymptotically better. Another group of spacings statistics are based on ordered spacings. In particular, if $D_{(n)} = \max_{1 \leq i \leq n} D_i$, then $R_n = (1 - D_{(n)})$ is referred to as the 'circular range', the shortest arc on the circumference containing all the observations. This is discussed in Rao (1969) and Laubscher and Rudolph (1968).

(ii) Tests based on empirical distribution functions. Given the random sample $\alpha_1, \ldots, \alpha_n$ on the circumference $[0, 1)$, one can define the empirical distribution function (in the usual linear sense) as

$$F_n(x) = \frac{\text{number of } \alpha_i \leq x}{n} \tag{3.5}$$

for $0 \leq x < 1$. The usual test statistics like the Kolmogorov–Smirnov statistic

$$K_n = \sqrt{n} \sup_{0 \leq x < 1} |F_n(x) - x| \tag{3.6}$$

or the Cramer–von Mises statistic

$$W_n^2 = n \int_0^1 (F_n(x) - x)^2 \, dx \tag{3.7}$$

do not have the required invariance property. Kuiper (1960) suggested the following variation of (3.6) which is rotationally invariant and hence usable with circular data. Let

$$D_n^+ = \sup_{0 \leq x < 1} (F_n(n) - x) \quad \text{and} \quad D_n^- = \sup_{0 \leq x < 1} (x - F_n(x))\,. \tag{3.8}$$

While the Kolmogorov–Smirnov statistic $K_n = \max(D_n^+, D_n^-)$, Kuiper's statistic

$$V_n = \sqrt{n}(D_n^+ + D_n^-)\,. \tag{3.9}$$

Its asymptotic null distribution (under the hypothesis (3.1) of uniformity) is given by (cf. Kuiper, 1960)

$$P(V_n \geq x) = \sum_{m=1}^{\infty} 2(4m^2x^2 - 1)\,\mathrm{e}^{-2m^2x^2} - \frac{8x}{3\sqrt{n}} \sum_{m=1}^{\infty} m^2(4m^2x^2 - 3)\,\mathrm{e}^{-2m^2x^2} + \mathrm{O}\left(\frac{1}{n}\right) \tag{3.10}$$

for $x \geq 0$. Stephens (1965) provides upper percentage points for small n. Watson (1961) defined an invariant version of (3.7), namely

$$U_n^2 = n \int_0^1 \left[F_n(x) - x - \int_0^1 (F_n(y) - y)\,\mathrm{d}y\right]^2 \mathrm{d}x \tag{3.11}$$

for use with circular data. Observe that U_n^2 is of the form of a variance while W_n^2 is like the second moment. The asymptotic null distribution is given by (refer to Watson, 1961)

$$\lim_{n\to\infty} P(U_n^2 > x) = 2 \sum_{m=1}^{\infty} (-1)^{m-1}\,\mathrm{e}^{-2m^2\pi^2x} \tag{3.12}$$

for $x \geq 0$.

(iii) Scan statistics and chi-square-type tests. Ajne (1968) suggested two test statistics based on the number of observations in a half-circle

$$N(\alpha) = \text{number of observations in } [\alpha, \alpha + \tfrac{1}{2})$$

for $0 \leq \alpha < 1$. One of them is to take

$$N = \sup_{0 \leq \alpha < 1} N(\alpha)\,, \tag{3.13}$$

the maximum number in any half-circle. As Rao (1969) and Bhattacharya and Johnson (1969) pointed out, this is related to a bivariate sign test suggested earlier by Hodges (1955). The exact null distribution of N is given by (cf. Ajne, 1968)

$$P(N \geq k) = \frac{(2k-n)}{2^{n+1}} \sum_{j=0}^{\infty} \binom{n}{k + j(2k-n)} \tag{3.14}$$

for $k \geq [n/2] + 1$ which reduces for $k > \frac{2}{3}n$ to the simpler expression

$$P(N \geq k) = \frac{2k-n}{2^{n+1}} \binom{n}{k}.$$

The asymptotic null distribution of

$$N^* = \frac{2}{\sqrt{n}}\left(N - \frac{n}{2}\right)$$

is given by

$$\lim_{n\to\infty} P(N^* > c) = 2c\sqrt{\frac{2}{\pi}} \sum_{j=0}^{\infty} \mathrm{e}^{-(2j+1)^2c^2/2} \tag{3.15}$$

for $c \geq 0$. Rothman (1972) considered the maximum number of observations in any arc of length p, $0 < p < 1$. These are related to the scan statistics. See, for instance, Naus (1982).

Another statistic for testing uniformity is obtained by averaging, i.e., by considering

$$A_n = \int_0^1 \left(N(\alpha) - \frac{n}{2}\right)^2 \mathrm{d}\alpha\,. \tag{3.16}$$

The asymptotic null distribution of A_n is given by

$$\lim_{n\to\infty} P(A_n > x) = \sum_{m=1}^{\infty} \frac{4(-1)^{m-1}}{\pi(2m-1)}\,\mathrm{e}^{-\pi^2(2m-1)^2x/2}$$

for $x \geq 0$. The statistic in (3.16) has been generalized in two different directions by Beran (1969a) and Rao (1972b). Rao (1972b) considered dividing the unit circumference into m (≥ 2) equal class intervals with the i-th interval being

$$I_i(\alpha) = \left[\alpha + \left(\frac{i-1}{m}\right), \alpha + \frac{i}{m}\right), \quad i = 1, \ldots, m\,.$$

Using the observed class frequencies $N_i(\alpha)$, $i = 1, \ldots, m$ in these intervals, one can construct a χ^2 statistic $\chi_n^2(\alpha) = \Sigma_{i=1}^m (N_i(\alpha) - n/m)^2/(n/m)$. This can be made invariant with respect to the choice of origin α by taking either the supremum over α or alternately by averaging as in (3.16), namely,

$$\chi_n^2 = \int_0^1 \chi_n^2(\alpha)\,\mathrm{d}\alpha\,. \tag{3.17}$$

The statistic A_n corresponds to χ_n^2 with $m = 2$. The asymptotic null distribution of χ_n^2 in (3.17) and a computational form for it are provided in Rao (1972b).

Beran (1969a) proposed the class of test statistics of the form

$$T_n = \int_0^1 \left[\sum_{i=1}^n (f(\alpha + \alpha_i) - 1)\right]^2 \mathrm{d}\alpha \tag{3.18}$$

where f is any probability density function on the circle. It can be verified that

A_n and U_n^2 defined in (3.16) and (3.11) are of this form. Beran (1969a) obtains the asymptotic distribution of this statistic under the null hypothesis (3.1) as well as under fixed alternatives and derives the approximate Bahadur slope. Tests based on T_n are best invariant against local alternatives (i.e., for small k) of the form

$$g(\alpha; k) = 1 + k(f(\alpha + \mu_0) - 1), \quad 0 \leq \alpha < 1, \ 0 \leq k \leq 1.$$

If we define

$$h(\theta) = 2 \sum_{p=1}^{\infty} \rho_p^2 \cos p\theta$$

where

$$\rho_p^2 = \left| \int_0^1 e^{ip\alpha} f(\alpha)\, d\alpha \right|^2,$$

then T_n can be rewritten as

$$T_n = \sum_{i=1}^{n} \sum_{j=1}^{n} h(\theta_i - \theta_j). \tag{3.19}$$

See for example Mardia (1975, pp. 190–191).

Some comparisons. Rao (1972a) compared the various tests of uniformity through the Bahadur efficiencies using von Mises–Fisher alternatives (cf. Equation (2.3)) with small concentration. For this situation Rayleigh's test based on the length $|R|$ of the resultant is uniformly most powerful invariant. These comparisons based on Bahadur efficiencies show that Watson's U_n^2 (cf. (3.11)) and Ajne's A_n (cf. (3.16)) tests have the same asymptotic efficiencies as the Rayleigh test, while Kuiper's test (cf. (3.9)) and the Hodges–Ajne N test (cf. (3.13)) have a lower asymptotic efficiency of $8/\pi$ or about 81% compared to the former group of test statistics. Symmetric spacings tests have poor asymptotic efficiencies but Monte Carlo comparisons show that for small samples, they have reasonable power compared to Rayleigh's test. Stephens (1969) compares Kuiper's V_n, Watson's U_n^2 and Ajne's A_n using Monte Carlo powers. His conclusions indicate that while these three tests are about equal in performance against unimodal alternatives, differences show up in testing uniformity against multimodal alternatives with Kuiper's V_n faring the best and U_n^2 and A_n following in that order.

Other one-sample tests. Schach (1969b) applies the Wilcoxon signed rank test for testing the hypothesis of symmetry. Rothman (1971) and Puri and Rao (1977) consider the problem of testing coordinate independence given bivariate data on a torus. There are a number of papers on the topic of developing measures of association for angular–angular or angular–linear data, some of them nonparametric. See Jupp and Mardia (1980) and other references contained there for measures of correlation for the angular–angular case, i.e., for observations on a torus. Fisher and Lee (1981) develop a U-statistic which is analogous to Kendall's τ for measuring angular linear association and derive its distribution.

Lenth (1981) utilizes a periodic version of the commonly used ψ functions to adapt robust M-estimation for use with directional data.

3.2. Two-sample tests

The usual nonparametric theory for the two-sample observations in the combined sample. See, e.g., Hajek and Sidak (1967). Since rank values on a circle depend on the starting point as well as the sense of rotation, such rank tests cannot be used with circular data. Schach (1969a) defines what may be called the 'circular ranks', which remain invariant under the following group of transformations. Let $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ denote the two independent random samples from F and G respectively, the hypothesis of interest being

$$H_0: \quad F(\alpha) = G(\alpha), \ 0 \leq \alpha < 1. \tag{3.20}$$

Let $(r_1, \ldots, r_m)$ denote the (linear) ranks of the first sample in the combined sample of $N = (m + n)$ observations in the usual fashion, and let

$$R = \{(r_1, \ldots, r_N) : (r_1, \ldots, r_N) \text{ is a permutation of integers } (1, \ldots, N)\}$$

be the space of rank vectors for the combined sample. Define groups of transformations $\{g\}$ (corresponding to changes in zero direction) and $\{h\}$ (corresponding to changes in sense of rotation), of R onto itself by

$$g: (r_1, \ldots, r_N) \to (r_1 + 1, \ldots, r_N + 1)$$

and

$$h: (r_1, \ldots, r_N) \to (N + 1 - r_N, \ldots, N + 1 - r_1)$$

where the components of the transformed vector are defined modulo N. Let $\mathcal{G}$ be the group of transformations $R \to R$ generated by $\{g\}$ and $\{h\}$. We may define circular ranks $(C_1, \ldots, C_m)$ of $(\alpha_1, \ldots, \alpha_m)$ as an equivalence class of $(r_1, \ldots, r_m)$ under the group $\mathcal{G}$. One can then define, corresponding to any linear rank test $T(r)$ based on linear ranks r, a circular rank test

$$T(c) = \sup_{g^* \in \mathcal{G}} T(g^*(r))$$

which will then possess the required invariance. Batschelet (1965) suggested such an invariant version of the Wilcoxon–Mann–Whitney statistic and provided a short table of critical values. Epplett (1982) pursues this further and obtains its asymptotic null distribution.

Let $F_m(x)$ and $G_n(x)$ denote the empirical distribution functions of the α's and β's, respectively. Define

$$D^+_{m,n} = \sup_{0\le x<1} [F_m(x) - G_n(x)] \tag{3.21}$$

and

$$D^-_{m,n} = \sup_{0\le x<1} [G_n(x) - F_m(x)]\,. \tag{3.22}$$

Let $\boldsymbol{r} = (r_1, \ldots, r_m)$ denote the linear ranks and

$$W_{m,n}(\boldsymbol{r}) = \sum_{i=1}^{m} r_i \tag{3.23}$$

the Wilcoxon test statistic. Then Epplett (1982) shows that the circular version

$$\begin{aligned} W_{m,n}(\boldsymbol{c}) &= \sup_{g^*\in\mathcal{G}} W_{m,n}(g^*(\boldsymbol{r})) \\ &= \max\{W_{m,n}(\boldsymbol{r}) + mnD^+_{m,n},\ m(N+1) - W_{m,n}(\boldsymbol{r}) + mnD^-_{m,n}\} \end{aligned} \tag{3.24}$$

and establishes that $\sqrt{mnN}\{W_{m,n}(\boldsymbol{c}) - \frac{1}{2}m(N+1)\}$ converges in distribution to that of $\sup_t|S(t)|$ where $S(t)$ is a Gaussian process with zero mean and covariance kernel

$$K(s,t) = \tfrac{1}{12} - \tfrac{1}{2}(t-s)(1-(t-s))$$

for $0 \le s \le t \le 1$. This test is shown to compare favorably with the two-sample Kuiper test (see Equation (3.25)) in terms of Bahadur efficiency. Through inclusion–exclusion, Epplett (1979) relates the exact probabilities for the circular statistic to those of the linear Wilcoxon statistic and provides a recurrence relation.

(i) Tests based on empirical distribution functions. Since the two-sample versions of the Kolmogorov–Smirnov and Cramer–von Mises statistics are not rotationally invariant, they are inappropriate for testing the hypothesis (3.20). Kuiper (1960) suggested the following two-sample variation of the Kolmogorov–Smirnov statistic:

$$V_{m,n} = (D^+_{m,n} + D^-_{m,n}) \tag{3.25}$$

where $D^+_{m,n}$ and $D^-_{m,n}$ are as defined in (3.21) and (3.22). Its asymptotic null distribution, properly normalized, is the same as that given in (3.10). Barr and Shudde (1973) show that

$$V_{m,n} = \sup_{g^*\in\mathcal{G}} D_{m,n}(g^*(\boldsymbol{r})) \tag{3.26}$$

where $D_{m,n} = \max(D^+_{m,n}, D^-_{m,n})$ is the usual two-sample Kolmogorov–Smirnov statistic. Similarly, a two-sample version of (3.11) was proposed by Watson (1962), namely

$$U^2_{m,n} = \frac{mn}{N}\int_0^1 \left[F_m(x) - G_n(x) - \int_0^1 (F_m(y) - G_n(y))\,\mathrm{d}H_N(y)\right]^2 \mathrm{d}H_N(x) \tag{3.27}$$

where $F_m(x)$ and $G_n(x)$ are the empirical distribution functions of α's and β's respectively and $H_N(x) = [mF_m(x) + nG_n(x)]/N$. The asymptotic null distribution of $U^2_{m,n}$ is again the same as that given in Equation (3.12).

(ii) Tests based on uniform scores. Beran (1969b) pointed out that two-sample tests for the hypothesis (3.20) can be obtained from tests of uniformity as follows: If $(r_1, \ldots, r_m)$ denote the (linear) ranks of the first sample in the combined sample, then define

$$u_i = r_i/N, \quad i = 1, \ldots, m\,, \tag{3.28}$$

called the 'uniform scores'. Under the null hypothesis $F = G$, these scores must be uniformly distributed on the circle of unit circumference. Thus any test of uniformity discussed in Section 3.1 can then be used on $\{u_i\}$ to test the hypothesis (3.20). A test which rejects this hypothesis for large values of $|\boldsymbol{R}_1|$, the length of the resultant of $\{u_i, i = 1, \ldots, m\}$ was proposed by Wheeler and Watson (1964). Mardia (1967) considered the statistic based on $|\boldsymbol{R}_1|$ in connection with a bivariate location problem. Mardia (1969) and Schach (1969a) discuss the asymptotic power and consistency properties of the Wheeler and Watson statistic. Schach (1969a) considers a general class of statistics of the form

$$T_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{m} h_N(u_i - u_j) \tag{3.29}$$

which corresponds to the two-sample adaptation of Beran's statistic T_n (cf. (3.19)) and shows that the asymptotic null distribution of $((N-1)T_{m,n}/mn)$ is the same as that of the one-sample statistic T_n if $n/N \to \lambda$, $0 < \lambda < 1$.

(iii) Tests based on spacing-frequencies. For the general two-sample problem, Holst and Rao (1980) investigate families of statistics based on the 'spacing-frequencies'. These are the frequencies of one sample, say β_j's, that fall in between the spacings made by the other sample. Thus the spacing-frequencies are defined by

$$S_i = \text{number of } \beta_j\text{'s in } [\alpha_{(i-1)}, \alpha_{(i)}), \quad i = 1, \ldots, m\,, \tag{3.30}$$

where $\{\alpha_{(i)}\}$ are ordered. Statistics based symmetrically on $\{S_i, i = 1, \ldots, m\}$ are clearly rotation invariant. Thus one may use test statistics of the form

$$T_{m,n} = \sum_{i=1}^{m} h_N(S_i) \tag{3.31}$$

for 'reasonable' functions $h_N(\cdot)$. The circular run test (cf. David and Barton, 1962) and a test suggested by Dixon (1940) based on $\Sigma_1^m S_i^2$, are special cases of this form. Holst and Rao (1980) show that under mild conditions on $h_N(\cdot)$, the statistics $T_{m,n}$ are asymptotically normal and that the Dixon test based on $\Sigma_1^m S_i^2$ is asymptotically locally most powerful among this class. Some further power comparisons and the special relevance of this class (3.31) to circular data problems are discussed in Rao and Mardia (1980). More recently, tests based on k-th order spacing-frequencies (for fixed finite k), i.e., on $S_i^{(k)}$ = the number of observations in $[\alpha_{(i-1)}, \alpha_{(i+k-1)})$, are considered in Rao and Schweitzer (1982) where it is shown that among tests symmetric in $\{S_i^{(k)}\}$, which can be used for circular distributions, $\Sigma_{i=1}^m S_i^{(k)2}$ is asymptotically locally most powerful.

3.3. *Multi-sample tests for circular data*

Given a random sample of size n_i, say $(\alpha_{i1}, \ldots, \alpha_{in_i})$ from the i-th population, $i = 1, \ldots, k$, one is interested in tests of homogeneity of these k populations. If these populations are unimodal, such tests of homogeneity (a) with respect to mean directions and (b) with respect to concentrations are proposed for large samples in Rao (1966) and are further discussed in Yoshimura (1978).

A k-sample analogue of Watson's $U^2_{m,n}$ (see Equation (3.27)) and its asymptotic null distribution are discussed in Maag (1966). Multisample analogues of other statistics like $V_{m,n}$ (Equation (3.25)) do not appear to have been considered in the literature. A test based on multiple runs on the circle is discussed in David and Barton (1962, pp. 119–136) and this has a long history. One can also construct tests based on 'uniform scores'

$$u_{ij} = r_{ij}/N \tag{3.32}$$

where $\{r_{ij}, u = 1, \ldots, n_i\}$ are the ranks of the i-th sample observations among the combined sample of $N = (n_1 + \cdots + n_k)$ observations. Mardia (1972b) considers a test based on the statistic

$$2 \sum_{i=1}^{k} (R_i^2/n_i) \tag{3.33}$$

where

$$R_i^2 = \Big(\sum_{j=1}^{n_i} \cos 2\pi u_{ij}\Big)^2 + \Big(\sum_{j=1}^{n_i} \sin 2\pi u_{ij}\Big)^2$$

is the squared length of the resultant for the uniform scores of the i-th sample. The statistic in (3.33) corresponds to the log likelihood ratio for testing homogeneity of mean directions of k von Mises–Fisher distributions and may also be thought of as an extension of the Wheeler and Watson test mentioned earlier. Mardia (1972b) also shows that the test in (3.33) when compared with the parametric competitor for the von Mises–Fisher distributions, has a Bahadur efficiency approaching one as the concentration parameter of the von Mises–Fisher distributions approaches zero.

4. Nonparametric methods for spherical data

Most of statistical methods developed for one-, two- or multi-sample spherical problems assume parametric models like the von Mises–Fisher distribution in (2.4) or other distributions with different properties. There has not been, in general, as much progress in developing useful nonparametric tests for spherical data. Though this is somewhat analogous to the situation in regard to nonparametric procedures for multivariate statistical analyses, it seems that there is scope for progress here along the lines of Puri and Sen (1971).

4.1. *One-sample problem-testing uniformity*

Beran (1968) considered a class of statistics similar to (3.18) for the more general problem of testing uniformity on a compact homogeneous space. He shows that this test is locally most powerful invariant and derives the asymptotic null distribution of the statistic. Specifically for the sphere, an analogue of (3.19) is based on

$$T_n = \frac{n}{4} - \frac{1}{\pi n} \sum_{i<j} \psi_{ij} \tag{4.1}$$

where ψ_{ij} is the smaller angle between the i-th and j-th observations (in polar coordinates) (α_i, β_i) and (α_j, β_j). Gates and Westcott (1980) discuss bounds on the distribution of the minimum interpoint angular distance $\psi = \min_{i,j} \psi_{ij}$ under the hypothesis of uniformity. Giné (1975) considers a class of invariant tests for uniformity based on Sobolev norms, which contains as special cases tests for uniformity on the circle, the sphere and the hemisphere (where the antipodes are identified) introduced earlier by Rayleigh, Watson (1961), Ajne (1968), Rao (1972b), Beran (1968) and Bingham (1964). Prentice (1978) follows along the lines of Giné (1975) and Beran (1968) to obtain a class of invariant tests for spheres and hemispheres in any dimension $p \geq 1$. Stephens (1966) tabulates the percentage points of three statistics which are useful in testing uniformity on the sphere against specified alternatives listed.

4.2. *Two-sample tests for spherical data*

Wellner (1979) considers a class of permutation tests for the two-sample problem when the data come from any arbitrary compact Riemannian manifold. Special cases of interest include tests for comparing two samples from the unit sphere in three dimensions or the hemisphere or the torus. The test statistics are the two-sample analogues of Gine's (1975) tests of uniformity. As with any permutation test, the idea is to calculate an invariant statistic $T_{m,n}$ for all $\binom{m+n}{n}$ choices of the first sample relabellings and reject the null hypothesis of identical distributions if the observed $T_{m,n}$ is 'too big' relative to the resulting conditional distribution (conditional on the pooled sample). Specified consistency properties and asymptotic distributions under the null and fixed alternatives are derived. Two-sample versions of the various test statistics of uniformity are discussed as examples.

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PART VIII

APPLICATIONS